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A conjecture arising naturally in the investigation of additivity of classical information capacity of quantum channels states that the maximal purity of outputs from a quantum channel, as measured by the p -norm, should be multiplicative with respect to the tensor product of channels. We disprove this conjecture for $p > 4.79$. The same example (with $p = \infty$) also disproves a conjecture for the multiplicativity of the injective norm of Hilbert space tensor products.

I. STATEMENT OF THE PROBLEM

A. Multiplicativity of output purity

In many applications of quantum information theory the entanglement of states or the capacity of channels appear as resources, which are needed to perform a task and are used up in the process. Therefore, it is natural to expect that certain entanglement or capacity measures should be *additive* in the sense that preparing two pairs of entangled particles should give us twice the entanglement of one pair and, similarly, using a channel twice doubles its capacity. However, such additivity properties have turned out to be notoriously difficult to prove, and in some cases folk conjectures claiming additivity have turned out to be wrong.

The purpose of this note is to provide a counterexample of this kind i.e. to show that a family of quantities, which had been conjectured to be additive in an earlier paper by the present authors [1], actually is not. The quantities considered all characterize the highest purity of the outputs of a channel. That is, if S is a completely positive map, taking density operators on a finite dimensional Hilbert space \mathcal{H}_1 to density operators on another finite dimensional Hilbert space \mathcal{H}_2 , and $1 < p < \infty$, we defined

$$\nu_p(S) = \sup \|S(\rho)\|_p, \quad (1)$$

where the supremum is over all input density operators, and $\|\rho\|_p = (\text{tr}|\rho|^p)^{1/p}$ is the standard p -norm. The conjecture in [1] was that $\log \nu_p$ is additive in the sense that

$$\nu_p(S_1 \otimes S_2) = \nu_p(S_1)\nu_p(S_2) \quad (2)$$

for arbitrary channels S_1, S_2 .

This conjecture was supported by some numerical evidence (the inequality “ \geq ” being trivial anyhow), and a proof for very noisy and almost noiseless channels in [1], as well as some depolarizing channels. Further supporting evidence was given by C. King [2,3]. The main application would probably have been in the limit $p \rightarrow 1$, where it would be the additivity of “maximal purity as measured by entropy”. This in turn is closely related [1,3,4] to the question of additivity of classical channel capacity, i.e., whether transmission of classical information over multiple quantum channels can sometimes be improved by using entangled signal states.

B. Injective tensor norm for Hilbert space vectors

The case $p = \infty$, i.e., when $\|\cdot\|_\infty$ is the ordinary operator norm, is implied by another additivity conjecture, namely for the *injective tensor norm* of Hilbert space vectors. For any vector $\Phi \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ we define this norm as

$$\mu_N(\Phi) = \sup |\langle \Phi, \phi_1 \otimes \cdots \otimes \phi_N \rangle|, \quad (3)$$

where the supremum is over all tuples of vectors $\phi_\alpha \in \mathcal{H}_\alpha$ with $|\phi_\alpha| = 1$. The conjectured property for this quantity was that $\mu_N(\Phi \otimes \Psi) = \mu_N(\Phi)\mu_N(\Psi)$, where $\Phi \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and $\Psi \in \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_N$ may be in different N -fold Hilbert space tensor products, and the supremum in $\mu_N(\Phi \otimes \Psi)$ is taken over unit vectors $\phi_\alpha \in \mathcal{H}_\alpha \otimes \mathcal{K}_\alpha$. Again, the inequality “ \geq ” and the case $N = 2$ are trivial. The connection with the previous problem is seen by writing S in Kraus form $S(\rho) = \sum_x A_x \rho A_x^*$. Then

$$\begin{aligned} \langle \varphi, S(|\phi\rangle\langle\phi|)\varphi \rangle &= \sum_x |\langle \Phi, A_x \phi \rangle|^2 \\ &= \sup_\psi \left| \sum_x \overline{\psi_x} \langle \varphi, A_x \phi \rangle \right|^2, \end{aligned} \quad (4)$$

where at the last equality we consider the $\langle \varphi, A_x \phi \rangle$ as the components of a Hilbert space vector, whose norm can also be written as the largest scalar product with a unit vector. Accordingly, the supremum in the last line is over all unit vectors ψ . Taking the supremum over the unit vectors ϕ and φ , too, we find that

$$\nu_\infty(S) = \mu_3(\tilde{A})^2, \quad (5)$$

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where \tilde{A} denotes the vector in a threefold Hilbert space tensor product with components $\langle h_j, A_x e_k \rangle$, where h_j and e_k are orthonormal bases of the appropriate spaces. In particular, since the tensor product of channels S corresponds to the tensor product of vectors \tilde{A} , the conjectured multiplicativity of μ_3 would imply the multiplicativity of ν_∞ . Conversely, the counterexample given to the latter disproves the multiplicativity of μ_N for all $N \geq 3$.

II. THE COUNTEREXAMPLE

We give an explicit example of a channel violating conjecture (2) for large values of p . It is the channel S on the $d \times d$ -matrices defined as

$$S(\rho) = \frac{1}{d-1} (\text{tr}(\rho) \mathbb{1} - \rho^T) \quad (6)$$

$$= \frac{1}{2(d-1)} \sum_{ij} (|i\rangle\langle j| - |j\rangle\langle i|)^* \rho (|i\rangle\langle j| - |j\rangle\langle i|). \quad (7)$$

Here ρ^T denotes the matrix transpose with respect to some fixed basis. In the first form it is easy to verify that S is linear and trace preserving, in the second it becomes clear that it is also completely positive. The equality of the two forms is a straightforward exercise. Further interesting properties are that S is hermitian with respect to the Hilbert-Schmidt scalar product $(A, B) \mapsto \text{tr}(A^* B)$ on $d \times d$ -matrices, and covariant for arbitrary unitary transformations in the sense that $S \circ \text{ad}_U = \text{ad}_{\bar{U}} \circ S$, $\text{ad}_U(X) = UXU^*$, where \bar{U} denotes the matrix element-wise complex conjugate of a unitary in the fixed basis. We remark that S is the dual of the state which provided the counterexample to the additivity of the relative entropy of entanglement for bipartite states in [5], i.e., that state, the normalized projection onto the Fermi subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$ ($d \geq 3$), is obtained by acting with S on one partner of a maximally entangled pair on $\mathbb{C}^d \otimes \mathbb{C}^d$.

Now $\rho \mapsto \|S(\rho)\|_p$ is a convex function, and hence takes its maximum on the extremal states. Therefore it suffices to take pure input states, for which

$$S(|\phi\rangle\langle\phi|) = \frac{1}{d-1} (\mathbb{1} - |\bar{\phi}\rangle\langle\bar{\phi}|). \quad (8)$$

Clearly, the p -norm is the same for all pure inputs, and we get

$$\nu_p(S) = (d-1)^{-(1-1/p)}. \quad (9)$$

On the other hand, let us consider $S \otimes S$ acting on a pure state Φ . Due to covariance we may take Φ in Schmidt diagonal form $\Phi = \sum_\alpha c_\alpha |\alpha\alpha\rangle$. Then with the reduced density operator $\rho = \sum_\alpha c_\alpha^2 |\alpha\rangle\langle\alpha|$ we get

$$\begin{aligned} S \otimes S(|\Phi\rangle\langle\Phi|) &= \\ &= \frac{1}{(d-1)^2} (\mathbb{1} - \mathbb{1} \otimes \rho - \rho \otimes \mathbb{1} + |\Phi\rangle\langle\Phi|). \end{aligned} \quad (10)$$

We now specialize further to *maximally entangled* $\Phi = \Phi_m$, i.e., all $c_\alpha = 1/\sqrt{d}$. and $\rho = (1/d)\mathbb{1}$, all terms in this expression commute, and the operator in parenthesis has one eigenvalue $(1 - 2/d)$ with multiplicity $(d^2 - 1)$ and a non-degenerate eigenvalue $(1 - 2/d + 1)$. From this we find, for $d = 3$:

$$\|S \otimes S(|\Phi_m\rangle\langle\Phi_m|)\|_p = \frac{1}{3} (1 + 2^{3-2p})^{1/p}. \quad (11)$$

If additivity were true, the following quantity should be negative, becoming zero upon maximization with respect to Φ :

$$\Delta(p, \Phi) = \log \|S \otimes S(|\Phi\rangle\langle\Phi|)\|_p - 2 \log \nu_p(S) \quad (12)$$

However, inserting for Φ the maximally entangled state we get

$$\Delta(p, \Phi_m) = \log \frac{4}{3} + \frac{1}{p} \log \left(\frac{1}{4} + 2^{1-2p} \right). \quad (13)$$

This quantity is plotted in Fig. 1. Thus additivity is violated for $p > p_0$, where the zero p_0 is numerically determined as $p_0 = 4.7823$. In particular, we get $\Delta(\infty, \Phi_m) = \log(4/3) > 0$, so additivity fails for $p = \infty$. Transcribed to the problem for injective Hilbert space norms, the counterexample is the unique antisymmetric vector Φ on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, tensored with itself. On the other hand, for $p \rightarrow 1$ we have $\Delta(p, \Phi_m) < 0$, so in this case, which is of main interest for classical channel capacity, the additivity conjecture survives.

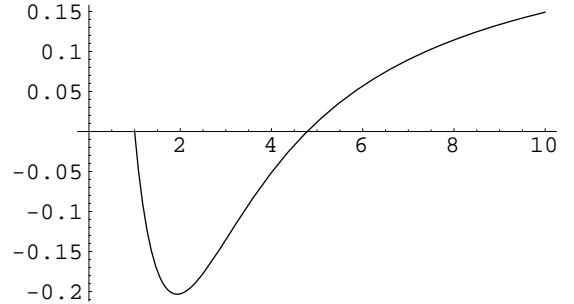


FIG. 1. $\Delta(p, \Phi_m)$ over p , as explained in the text.

The boundary point p_0 cannot be improved by choosing another vector Φ , i.e., the maximizing Φ in the definition of $\nu_p(S \otimes S)$ jumps discontinuously from a product state to a maximally entangled state, as p increases beyond p_0 . This is seen by plotting Δ for fixed p over the Schmidt parameters of Φ . There is little difference between the plots for $p = 4$ and $p = 5$ except that the global maximum switches from corners (Φ a product vector) to the center ($\Phi = \Phi_m$).

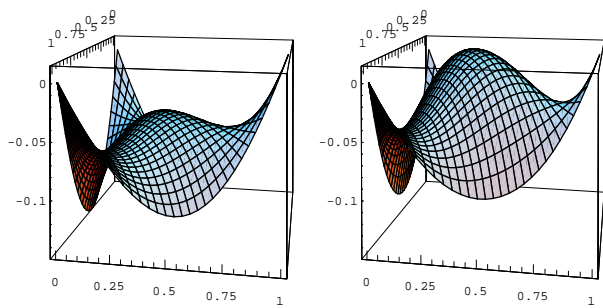


FIG. 2. $\Delta(p, \Phi)$ for $d = 3$ over the Schmidt parameters c_1^2 and c_2^2 of Φ , for $p = 4$ (left) and for $p = 5$ (right).

III. CONCLUDING REMARKS

We have seen that multiplicativity of maximal purity depends crucially on how we measure purity: if we use p -norms for large p , it fails. But the conjecture remains open for small p , and in particular for purity as measured by entropy. There seems to be no simple modification of the example given here to improve the critical value of p . On the other hand, it is hardly expected to be optimal.

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